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Singular Problems: An Upper and Lower Solution Approach

Donal O'Regan

Department of Mathematics, National University of Ireland, Galway, Ireland

and

Ravi P. Agarwal

Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260

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This paper presents an upper and lower solution theory for singular boundary value problems where the nonlinearity may be singular in the dependent variable. In addition our nonlinearity is allowed to change sign. © 2000 Academic Press

1. INTRODUCTION

This paper presents existence results for singular second order differential equations of the form

$$(py')' + p(t)q(t)f(t, y, py') = 0, \qquad 0 < t < 1 \tag{1.1}$$

with mixed boundary data

$$\lim_{t \to 0^+} p(t)y'(t) = y(1) = 0. \tag{1.2}$$

Here $p \in C[0,1] \cap C^1(0,1)$, $q \in C(0,1)$, $p > 0$ on $(0, 1)$, $q > 0$ on $(0, 1)$, and f is allowed to change sign. We note that $\frac{1}{p}$ is *not* necessarily in

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$L^1[0,1]$. In addition f may not be a Carathéodory function because of the singular behaviour of the y variable. In this paper we present an upper and lower solution type theory for the boundary value problem (1.1), (1.2). We note that some of the ideas here were motivated by recent papers of Agarwal and O'Regan [1–4], and Kannan and O'Regan [10]. Finally we remark that the ideas in this paper could be extended so that boundary data different from (1.2) could be discussed.

2. SINGULAR PROBLEMS

In this section we first discuss the singular problem

$$\begin{cases} (py')' + p(t)q(t)f(t,y) = 0, \quad 0 < t < 1 \\ \lim_{t\to 0^+} p(t)y'(t) = 0 \\ y(1) = 0. \end{cases} \tag{2.1}$$

We are interested in nonnegative solutions (in fact solutions y with $y > 0$ on [0,1)). Our main result involves approximating (2.1) by a sequence of nonsingular problems each of which has a lower solution ρ_n and an upper solution β_n. Using the Schauder fixed point theorem we establish the existence of a solution which lies between the lower and upper solution for each approximating problem. The Arzela–Ascoli theorem then completes the proof. For our first result we will suppose the following conditions are satisfied:

$$p \in C[0,1] \cap C^1(0,1) \qquad \text{with } p > 0 \text{ on } (0,1) \tag{2.2}$$

$$q \in C(0,1) \qquad \text{with } q > 0 \text{ on } (0,1) \tag{2.3}$$

$$\int_0^1 p(s)q(s)ds < \infty \qquad \text{and} \qquad \int_0^1 \frac{1}{p(t)} \int_0^t p(s)q(s)ds\,dt < \infty \tag{2.4}$$

$$f : [0,1] \times (0,\infty) \to \mathbf{R} \text{ is continuous} \tag{2.5}$$

$$\begin{aligned} &|f(t,y)| \le g(y) + h(y) \qquad \text{on } [0,1]\times(0,\infty) \text{ with } g > 0 \\ &\text{continuous and nonincreasing on } (0,\infty),\ h \ge 0 \text{ continuous} \\ &\text{on } [0,\infty),\ \text{ and } \tfrac{h}{g} \text{ nondecreasing on } (0,\infty) \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\text{let } n \in \{3,4,\dots\} \text{ and associated with each } n \text{ we have a constant} \\ &\rho_n \text{ such that } \{\rho_n\} \text{ is a nonincreasing sequence with } \lim_{n\to\infty} \rho_n = 0 \\ &\text{and such that for } \tfrac{1}{n} \le t \le 1 \text{ we have } p(t)q(t)f(t,\rho_n) \ge 0 \end{aligned} \tag{2.7}$$

$$\begin{aligned}&\text{there exists a function } \alpha \in C[0,1] \cap C^2(0,1)\\ &\text{with } p\alpha' \in AC[0,1],\ \lim_{t\to 0^+} p(t)\alpha'(t) = \alpha(1) = 0,\ \alpha > 0 \text{ on } [0,1)\\ &\text{such that for each } n \in \{3,4,\ldots\} \text{ we have } \quad p(t)q(t)f(t,y) +\\ &(p(t)\alpha'(t))' > 0 \text{ for } (t,y) \in \left[\tfrac{1}{n}, 1\right) \times \{y \in (0,\infty) : y < \alpha(t)\}\\ &\text{and } p(t)q(t)f\left(\tfrac{1}{n}, y\right) + (p(t)\alpha'(t))' > 0 \text{ for}\\ &(t,y) \in \left(0, \tfrac{1}{n}\right) \times \{y \in (0,\infty) : y < \alpha(t)\}\end{aligned} \tag{2.8}$$

$$\begin{aligned}&\text{for each } n \in \{3,4,\ldots\} \text{ there exists a function}\\ &\beta_n \in C[0,1] \cap C^2(0,1),\ p\beta_n' \in AC[0,1], \text{ with}\\ &\lim_{t\to 0^+} p(t)\beta_n'(t) \le 0,\ \beta_n(t) \ge \rho_n \text{ for } t \in [0,1] \text{ and with}\\ &p(t)q(t)f(t,\beta_n(t)) + (p(t)\beta_n'(t))' \le 0 \text{ for } t \in \left[\tfrac{1}{n}, 1\right)\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\text{for each } n \in \{3,4,\ldots\} \text{ we have } p(t)q(t)f\left(\tfrac{1}{n}, \beta_n(t)\right)\\ &+(p(t)\beta_n'(t))' \le 0 \text{ for } t \in \left(0, \tfrac{1}{n}\right)\end{aligned} \tag{2.10}$$

and

$$\max\left\{\sup_{t\in[0,1]} \beta_n(t) : n \in \{3,4,\ldots\}\right\} < \infty. \tag{2.11}$$

THEOREM 2.1. *Suppose* (2.2)–(2.11) *hold. In addition assume*

$$\int_0^1 \frac{1}{p(s)} \int_0^s p(x)q(x)g(\alpha(x))dx\,ds < \infty \tag{2.12}$$

is satisfied. Then (2.1) *has a solution* y (*here* $y \in C[0,1] \cap C^2(0,1)$ *with* $py' \in AC[0,1]$) *with* $y(t) \ge \alpha(t)$ *for* $t \in [0,1]$.

Proof. Fix $n \in N^+ = \{3,4,\ldots\}$. Consider the boundary value problem

$$\begin{aligned}&(py')' + pq\,f^\star(t,y) = 0, \qquad 0 < t < 1\\ &\lim_{t\to 0^+} p(t)y'(t) = 0\\ &y(1) = \rho_n,\end{aligned} \tag*{(2.13)^n}$$

where

$$f^\star(t,y)=\begin{cases} f\left(\frac{1}{n},\beta_n(t)\right)+r(\beta_n(t)-y), & y\ge\beta_n(t) \text{ and } 0\le t\le\frac{1}{n}\\ f(t,\beta_n(t))+r(\beta_n(t)-y), & y\ge\beta_n(t) \text{ and } \frac{1}{n}\le t\le 1\\ f\left(\frac{1}{n},y\right), & \rho_n\le y\le\beta_n(t) \text{ and } 0\le t\le\frac{1}{n}\\ f(t,y), & \rho_n\le y\le\beta_n(t) \text{ and } \frac{1}{n}\le t\le 1\\ f(t,\rho_n)+r(\rho_n-y), & y\le\rho_n \text{ and } \frac{1}{n}\le t\le 1\\ f\left(\frac{1}{n},\rho_n\right)+r(\rho_n-y), & y\le\rho_n \text{ and } 0\le t\le\frac{1}{n}\end{cases}$$

and $r:\mathbf{R}\to[-1,1]$ is the radial retraction defined by

$$r(x)=\begin{cases} x, & |x|\le 1\\ \frac{x}{|x|}, & |x|>1.\end{cases}$$

It is easy to see, via the Schauder fixed point theorem (see [11]), that $(2.13)^n$ has a solution $y_n\in C[0,1]\cap C^2(0,1)$ with $py_n'\in AC[0,1]$. First we show

$$y_n(t)\ge\rho_n \qquad \text{for } t\in[0,1]. \tag{2.14}$$

Suppose (2.14) is not true. Then $y_n-\rho_n$ has a negative absolute minimum at $t_0\in[0,1)$ (note $y_n(1)-\rho_n=0$). First let us take the case $t_0\in(0,1)$. Then $y_n'(t_0)=0$ and $(py_n')'(t_0)\ge 0$ (note $y_n(t_0)-\rho_n<0$). However,

$$\begin{aligned}(py_n')'(t_0)&=-p(t_0)q(t_0)f^\star(t_0,y_n(t_0))\\ &=\begin{cases}-p(t_0)q(t_0)[f(t_0,\rho_n)+r(\rho_n-y_n(t_0))] & \text{if } \frac{1}{n}\le t_0<1\\ -p(t_0)q(t_0)\left[f\left(\frac{1}{n},\rho_n\right)+r(\rho_n-y_n(t_0))\right] & \text{if } 0<t_0\le\frac{1}{n}\end{cases}\\ &<0,\end{aligned}$$

a contradiction. It remains to consider the case $t_0=0$. Notice $\lim_{t\to 0^+}p(t)[y_n-\rho_n]'(t)=0$. Also since $y_n(0)-\rho_n<0$ there exists $\delta>0$ with $y_n(s)-\rho_n<0$ for $s\in[0,\delta]$. Thus for $t\in(0,\delta)$,

$$p(t)(y_n-\rho_n)'(t)=-\int_0^t p(s)q(s)f^\star(s,y_n(s))\,ds<0,$$

and this contradicts the fact that $y_n-\rho_n$ has a negative absolute maximum at $t_0=0$. Thus (2.14) holds.

Next we show

$$y_n(t)\le\beta_n(t) \qquad \text{for } t\in[0,1]. \tag{2.15}$$

If (2.15) is not true then $y_n-\beta_n$ would have a positive absolute maximum at say $t_0\in[0,1)$. First let us take $t_0\in(0,1)$, in which case $p(y_n-\beta_n)'(t_0)=0$ and $(p(y_n-\beta_n)')'(t_0)\le 0$. There are two cases to consider, namely $t_0\in\left[\frac{1}{n},1\right)$ and $t_0\in\left(0,\frac{1}{n}\right)$.

Case (i). $t_0 \in \left[\frac{1}{n}, 1\right)$. Then since $y_n(t_0) > \beta_n(t_0)$ we have using (2.9) that

$$\begin{aligned}(p(y_n - \beta_n)')'(t_0) &= -p(t_0)q(t_0)f^\star(t_0, y_n(t_0)) - (p\beta_n')'(t_0) \\ &= -p(t_0)q(t_0)[f(t_0, \beta_n(t_0)) \\ &\qquad + r(\beta_n(t_0) - y_n(t_0))] - (p\beta_n')'(t_0) \\ &> 0,\end{aligned}$$

a contradiction.

Case (ii). $t_0 \in \left(0, \frac{1}{n}\right)$. Then (2.10) gives

$$\begin{aligned}(p(y_n - \beta_n)')'(t_0) = -p(t_0)q(t_0)&\left[f\left(\frac{1}{n}, \beta_n(t_0)\right) + r(\beta_n(t_0) - y_n(t_0))\right] \\ &- (p\beta_n')'(t_0) > 0,\end{aligned}$$

a contradiction.

It remains to consider the case $t_0 = 0$. Now

$$\lim_{t \to 0^+} p(t)[y_n - \beta_n]'(t) = -\lim_{t \to 0^+} p(t)\beta_n'(t) \geq 0,$$

which is a contradiction unless $\lim_{t\to 0^+} p(t)\beta_n'(t) = 0$. Suppose $\lim_{t\to 0^+} p(t)\beta_n'(t) = 0$. Now $y_n(0) > \beta_n(0)$ guarantees that there exists $\mu > 0$, $\mu < \frac{1}{n}$ with

$$y_n(s) - \beta_n(s) > 0 \qquad \text{for } s \in (0, \mu).$$

Thus for $t \in (0, \mu)$ we have

$$\begin{aligned}p(y_n - \beta_n)'(t) &= -\int_0^t [p(s)q(s)f^\star(s, y_n(s)) + (p\,\beta_n')'(s)]\,ds \\ &= -\int_0^t \left[p(s)q(s)\left\{f\left(\frac{1}{n}, \beta_n(s)\right) + r(\beta_n(s) - y_n(s))\right\}\right. \\ &\qquad\qquad \left. + (p\,\beta_n')'(s)\right] ds \\ &> -\int_0^t \left[p(s)q(s)f\left(\frac{1}{n}, \beta_n(s)\right) + (p\,\beta_n')'(s)\right] ds \\ &\geq 0.\end{aligned}$$

That is, $p(y_n - \beta_n)'(t) > 0$ for $t \in (0, \mu)$, which is a contradiction. Thus (2.15) holds. In particular

$$y_n(t) \leq a_0 = \max\left\{\sup_{t\in[0,1]} \beta_n(t) : n \in \{3, 4, \ldots\}\right\} \qquad \text{for } t \in [0, 1]. \quad (2.16)$$

Next we obtain a sharper lower bound on y_n, namely we will show

$$y_n(t) \ge \alpha(t) \qquad \text{for } t \in [0,1]. \tag{2.17}$$

Suppose (2.17) is not true. Then $y_n - \alpha$ has a negative absolute minimum at $t_1 \in [0,1)$ (note $y_n(1) - \alpha(1) = \rho_n > 0$). First let us take $t_1 \in (0,1)$. Then $p(y_n-\alpha)'(t_1) = 0$ and $(p(y_n-\alpha)')'(t_1) \ge 0$. There are two cases to consider, namely $t_1 \in \left[\frac{1}{n}, 1\right)$ and $t_1 \in \left(0, \frac{1}{n}\right)$.

Case (i). $t_1 \in \left[\frac{1}{n}, 1\right)$. Now $0 < y_n(t_1) < \alpha(t_1)$, $\rho_n \le y_n(t_1) \le \beta_n(t_1)$, and (2.8) imply

$$\begin{aligned}(p(y_n-\alpha)')'(t_1) &= -[p(t_1)q(t_1)f^\star(t_1, y_n(t_1)) + (p\,\alpha')'(t_1)]\\ &= -[p(t_1)q(t_1)f(t_1, y_n(t_1)) + (p\,\alpha')'(t_1)]\\ &< 0,\end{aligned}$$

a contradiction.

Case (ii). $t_1 \in \left(0, \frac{1}{n}\right)$. Again (2.8) implies

$$\begin{aligned}(p(y_n-\alpha)')'(t_1) &= -[p(t_1)q(t_1)f^\star(t_1, y_n(t_1)) + (p\,\alpha')'(t_1)]\\ &= -\left[p(t_1)q(t_1)f\left(\frac{1}{n}, y_n(t_1)\right) + (p\,\alpha')'(t_1)\right]\\ &< 0,\end{aligned}$$

a contradiction.

It remains to consider the case $t_1 = 0$. Notice $\lim_{t\to 0^+} p(t)[y_n - \alpha]'(t) = 0$. Now there exists $\mu > 0$, $\mu < \frac{1}{n}$ with $0 < y_n(s) < \alpha(s)$ for $t \in [0,\mu]$ (also note $\rho_n \le y_n(s) \le \beta_n(s)$ for $s \in [0,\mu]$). Thus for $t \in (0,\mu)$ we have

$$\begin{aligned}p(y_n-\alpha)'(t) &= -\int_0^t [p(s)q(s)f^\star(s, y_n(s)) + (p\,\alpha')'(s)]\,ds\\ &= -\int_0^t \left[p(s)q(s)f\left(\frac{1}{n}, y_n(s)\right) + (p\,\alpha')'(s)\right]ds\\ &< 0.\end{aligned}$$

This contradicts the fact that $y_n - \alpha$ has a negative absolute minimum at $t_1 = 0$. Thus (2.17) is true.

Remark 2.1. It is easy to check directly, using the above type argument and (2.8), that $\alpha(t) \le \beta_n(t)$ for $t \in [0,1]$.

We shall now obtain a solution of (2.1) by means of the Arzela–Ascoli Theorem, as a limit of solutions of $(2.13)^n$. To this end we will show

$$\{y_n\}_{n\in N^+} \text{ is a bounded, equicontinuous family on } [0,1]. \tag{2.18}$$

To show equicontinuity notice

$$
\begin{aligned}
|f^\star(t, y_n(t))| &\le g(y_n(t))\left\{1+\frac{h(y_n(t))}{g(y_n(t))}\right\} \\
&\le g(\alpha(t))\left\{1+\frac{h(a_0)}{g(a_0)}\right\} \qquad \text{for } t \in (0,1).
\end{aligned}
$$

This together with the differential equation gives

$$
|y_n'(t)| \le \frac{1}{p(t)}\left\{1+\frac{h(a_0)}{g(a_0)}\right\}\int_0^t p(s)q(s)g(\alpha(s))\,ds \qquad \text{for } t \in (0,1)
$$

and this together with (2.12) establishes (2.18).

The Arzela–Ascoli Theorem guarantees the existence of a subsequence N_0 of N^+ and a function $y \in C[0,1]$ with y_n converging uniformly on [0,1] to y as $n \to \infty$ through N_0. Also $y(1) = 0$ and $y(t) \ge \alpha(t)$ for $t \in [0,1]$. Fix $t \in (0,1)$ and let $n_1 \in N_0$ be such that $1/n_1 < t < 1$. Let $N_1 = \{n \in N_0 : n \ge n_1\}$. Now y_n, $n \in N_1$, satisfies the integral equation

$$
\begin{aligned}
y_n(t) = y_n(0) &- \int_0^{\frac{1}{n}} \frac{1}{p(x)}\int_0^x p(s)q(s)f\left(\frac{1}{n}, y_n(s)\right)ds\,dx \\
&- \int_0^t \frac{1}{p(x)}\chi_{\left[\frac{1}{n},t\right]}(x)\left[\int_0^{\frac{1}{n}} p(s)q(s)f\left(\frac{1}{n}, y_n(s)\right)ds \right.\\
&\left. + \int_0^x p(s)q(s)f(s, y_n(s))\chi_{\left[\frac{1}{n},x\right]}(s)\,ds\right]dx.
\end{aligned}
$$

For $s \in [0,t]$ we have $f(s, y_n(s)) \to f(s, y(s))$ uniformly on compact subsets of $[0,t] \times (0, a_0]$, so letting $n \to \infty$ through N_1 gives

$$
y(t) = y(0) - \int_0^t \frac{1}{p(x)}\int_0^x p(s)q(s)f(s, y(s))\,ds\,dx. \tag{2.19}
$$

We can do this argument for each $t \in (0,1)$.

Remark 2.2. Notice to apply this step we need only $\int_0^a \frac{1}{p(x)}\int_0^x p(s)q(s)$ $g(\alpha(s))\,ds\,dx < \infty$ for any $a \in (0,1)$. This is automatically satisfied since (2.4) holds and $\alpha(s) > 0$ for $s \in [0,a]$. As a result (2.12) is *not* needed in this step.

From the integral equation (2.19) we see that $(py')'(t) + p(t)q(t)$ $f(t, y(t)) = 0$, $0 < t < 1$ and $\lim_{t\to 0^+} p(t)y'(t) = 0$. ∎

Remark 2.3. If in (2.7) we replace $\frac{1}{n} \le t \le 1$ with $0 \le t \le 1 - \frac{1}{n}$ then one would replace (2.8), (2.9), and (2.10) in Theorem 2.1 with

$$\begin{array}{l} \text{there exists a function } \alpha \in \mathrm{C}[0,1] \cap \mathrm{C}^2(0,1) \text{ with } \mathrm{p}\,\alpha' \in \mathrm{AC}[0,1], \\ \lim_{t\to 0^+} p(t)\alpha'(t) = \alpha(1) = 0,\ \alpha > 0 \text{on } [0,1) \text{ such that for each} \\ n \in \{3,4,\dots\} \text{ we have } p(t)q(t)f(t,y) + (p(t)\alpha'(t))' > 0 \text{ for} \\ (t,y) \in \left(0, 1-\frac{1}{n}\right] \times \{y \in (0,\infty) : y < \alpha(t)\} \text{ and} \\ p(t)q(t)f\left(1-\frac{1}{n}, y\right) + (p(t)\alpha'(t))' > 0 \text{ for} \\ (t,y) \in \left(1-\frac{1}{n}, 1\right) \times \{y \in (0,\infty) : y < \alpha(t)\} \end{array} \tag{2.20}$$

$$\begin{array}{l} \text{for each } n \in \{3,4,\dots\} \text{ there exists a function} \beta_n \in C[0,1] \cap C^2(0,1), \\ p\,\beta_n' \in AC[0,1], \text{with } \lim_{t\to 0^+} p(t)\beta_n'(t) \le 0,\ \beta_n(t) \ge \rho_n \text{ for } t \in [0,1] \\ \text{and with } p(t)q(t)f(t,\beta_n(t)) + (p(t)\beta_n'(t))' \le 0 \text{ for } t \in \left(0, 1-\frac{1}{n}\right] \end{array} \tag{2.21}$$

and

$$\begin{array}{l} \text{for each } n \in \{3,4,\dots\} \text{ we have } p(t)q(t)f\left(1-\frac{1}{n}, \beta_n(t)\right) \\ \qquad +(p(t)\beta_n'(t))' \le 0 \text{ for } t \in \left(1-\frac{1}{n}, 1\right). \end{array} \tag{2.22}$$

Remark 2.4 If in (2.7) we replace $\frac{1}{n} \le t \le 1$ with $\frac{1}{n} \le t \le 1 - \frac{1}{n}$ then one would replace (2.8), (2.9), and (2.10) in Theorem 2.1 with

$$\begin{array}{l} \text{there exists a function } \alpha \in C[0,1] \cap C^2(0,1) \text{ with } p\,\alpha' \in AC[0,1], \\ \lim_{t\to 0^+} p(t)\alpha'(t) = \alpha(1) = 0,\ \alpha > 0 \text{ on } [0,1) \text{ such that for each} \\ n \in \{3,4,\dots\} \text{ we have } p(t)q(t)f(t,y) + (p(t)\alpha'(t))' > 0 \text{ for} \\ (t,y) \in \left[\frac{1}{n}, 1-\frac{1}{n}\right] \times \{ y \in (0,\infty) : y < \alpha(t)\} \text{ and} \\ p(t)q(t)f\left(\frac{1}{n}, y\right) + (p(t)\alpha'(t))' > 0 \text{ for} \\ (t,y) \in \left(0, \frac{1}{n}\right) \times \{ y \in (0,\infty) : y < \alpha(t)\} \text{ and} \\ p(t)q(t)f\left(1-\frac{1}{n}, y\right) + (p(t)\alpha'(t))' > 0 \text{ for} \\ (t,y) \in \left(1-\frac{1}{n}, 1\right) \times \{y \in (0,\infty) : y < \alpha(t)\} \end{array} \tag{2.23}$$

$$\begin{array}{l} \text{for each } n \in \{3,4,\dots\} \text{ there exists a function } \beta_n \in C[0,1] \cap C^2(0,1), \\ p\,\beta_n' \in AC[0,1], \text{with } \lim_{t\to 0^+} p(t)\beta_n'(t) \le 0,\ \beta_n(t) \ge \rho_n \text{ for } t \in [0,1] \\ \text{and with } p(t)q(t)f(t,\beta_n(t)) + (p(t)\beta_n'(t))' \le 0 \text{ for } t \in \left[\frac{1}{n}, 1-\frac{1}{n}\right] \end{array} \tag{2.24}$$

$$\begin{array}{l} \text{for each } n \in \{3,4,\dots\} \text{ we have } p(t)q(t)f\left(\frac{1}{n}, \beta_n(t)\right) + (p(t)\beta_n'(t))' \le 0 \\ \text{for } t \in \left(0, \frac{1}{n}\right) \end{array} \tag{2.25}$$

and

$$\begin{array}{l} \text{for each } n \in \{3,4,\dots\} \text{ we have } p(t)q(t)f\left(1-\frac{1}{n}, \beta_n(t)\right) \\ \qquad +(p(t)\beta_n'(t))' \le 0 \text{ for } t \in \left(1-\frac{1}{n}, 1\right). \end{array} \tag{2.26}$$

Remark 2.5 If in (2.7) we replace $\frac{1}{n} \le t \le 1$ with $0 \le t \le 1$ then it is easy to see that (2.10) is not needed in the statement of Theorem 2.1 provided we assume $p(t)q(t)f(t, \beta_n(t)) + (p(t)\beta_n'(t))' \le 0$ for $t \in (0, 1)$. A similar remark applies to α. In this case we define $f^\star$ as

$$f^\star(t, y) = \begin{cases} f(t, \beta_n(t)) + r(\beta_n(t) - y), & y \ge \beta_n(t) \\ f(t, y), & \rho_n \le y \le \beta_n(t) \\ f(t, \rho_n) + r(\rho_n - y), & y \le \rho_n. \end{cases}$$

It is worth remarking that the only place we needed assumption (2.12) was in proving (2.18). It is possible to put other conditions on p, q, and f to guarantee that (2.18) holds.

THEOREM 2.2. *Suppose* (2.2)–(2.11) *hold. In addition assume*

$$\int_0^1 \frac{ds}{p(s)} < \infty \tag{2.27}$$

and

$$\begin{aligned} &\textit{for any } R > 0,\ \tfrac{1}{g} \textit{ is differentiable on } (0, R] \textit{ with } g' < 0 \\ &\textit{a.e. on } (0, R],\ g'/g^2 \in L^1[0, R],\ \textit{and } \textstyle\int_0^\infty (|g'(t)|^{1/2}/g(t))\, dt = \infty \end{aligned} \tag{2.28}$$

are satisfied. Then (2.1) *has a solution* y (*here* $y \in C[0, 1] \cap C^2(0, 1)$ *with* $p\, y' \in AC[0, 1]$) *with* $y(t) \ge \alpha(t)$ *for* $t \in [0, 1]$.

Proof. The proof is essentially the same as in Theorem 2.1 except to prove (2.18) we use the argument in [12, p. 34]. ∎

EXAMPLE 2.1. The boundary value problem

$$\begin{aligned} &(t^3 y')' + t^2\left(\tfrac{1}{\sqrt{y}} - \mu\right) = 0, \qquad 0 < t < 1 \\ &\lim_{t\to 0^+} t^3 y'(t) = 0 \\ &y(1) = 0, \qquad \mu > 0 \end{aligned} \tag{2.29}$$

has a solution $y \in C[0, 1] \cap C^2(0, 1)$ with $p\, y' \in AC[0, 1]$ and $y(t) > 0$ for $t \in [0, 1)$.

We will apply Theorem 2.1. Take $p(t) = t^3$, $q(t) = \frac{1}{t}$, $f(t, y) = 1/\sqrt{y} - \mu$, $g(y) = 1/\sqrt{y}$, and $h(y) = \mu$. Notice (2.2)–(2.6) are satisfied. Choose $n_0 \in \{1, 2, \ldots\}$ so that $n_0 \ge \mu^2$ and let

$$\rho_n = \frac{1}{n + n_0}.$$

Now (2.7) is true since

$$p(t)q(t)f(t, \rho_n) = t^2\left[(n + n_0)^{\frac{1}{2}} - \mu\right] \ge t^2\left(n_0^{\frac{1}{2}} - \mu\right) \ge 0 \qquad \text{for } t \in (0, 1).$$

Next let

$$\alpha(t) = b_0(1-t), \qquad \text{where } b_0 \geq 0 \text{ is chosen so that } 3\,b_0^{\frac{3}{2}} + \mu\, b_0^{\frac{1}{2}} < 1.$$

Now $\alpha(1) = 0$, $\lim_{t\to 0^+} t^3\,\alpha'(t) = 0$ and for $(t, y) \in \left[\frac{1}{n}, 1\right) \times \{y \in (0,\infty) : y < \alpha(t)\}$,

$$\begin{aligned} p(t)q(t)f(t,y) + (p\,\alpha')'(t) &= t^2\left(\frac{1}{\sqrt{y}} - \mu\right) - 3\,b_0 t^2 \\ &\geq t^2\left(\frac{1}{\sqrt{b_0(1-t)}} - \mu\right) - 3\,b_0\,t^2 \\ &= t^2\left(\frac{1}{\sqrt{b_0}\sqrt{(1-t)}} - \mu - 3\,b_0\right) \\ &\geq t^2\left(\frac{1}{\sqrt{b_0}} - \mu - 3\,b_0\right) \\ &= \frac{t^2}{\sqrt{b_0}}\left(1 - \mu\sqrt{b_0} - 3\,b_0^{\frac{3}{2}}\right) > 0. \end{aligned}$$

Also for $(t, y) \in \left(0, \frac{1}{n}\right) \times \{y \in (0,\infty) : y < \alpha(t)\}$, we have

$$\begin{aligned} p(t)q(t)f\left(\frac{1}{n}, y\right) + (p\,\alpha')'(t) &= \frac{1}{n^2}\left(\frac{1}{\sqrt{y}} - \mu\right) - 3\,b_0\,t^2 \\ &\geq t^2\left(\frac{1}{\sqrt{y}} - \mu\right) - 3\,b_0 t^2 \\ &\geq \frac{t^2}{\sqrt{b_0}}\left(1 - \mu\sqrt{b_0} - 3\,b_0^{\frac{3}{2}}\right) > 0. \end{aligned}$$

Thus (2.8) holds. Let

$$\beta_n(t) = \frac{1}{\mu^2} + \rho_n.$$

Clearly $\beta_n(t) \geq \rho_n$, $\lim_{t\to 0^+} p(t)\beta_n'(t) = 0$, and for $t \in \left[\frac{1}{n}, 1\right)$ we have

$$p(t)q(t)f(t, \beta_n(t)) + (p\,\beta_n')'(t) = t^2\left(\frac{1}{\sqrt{\beta_n(t)}} - \mu\right) \leq t^2(\mu - \mu).$$

Thus (2.9) holds. Also (2.10) is clear since if $t \in \left(0, \frac{1}{n}\right)$ we have

$$p(t)q(t)f\left(\frac{1}{n}, \beta_n(t)\right) + (p\,\beta_n')'(t) = \frac{1}{n^2}\left(\frac{1}{\sqrt{\beta_n(t)}} - \mu\right) \leq \frac{1}{n^2}(\mu - \mu).$$

Finally note (2.11) and (2.12) hold since

$$\begin{aligned}\int_0^1 \frac{1}{t^3}\int_0^t s^2\, g(\alpha(s))ds\,dt &= \frac{1}{\sqrt{b_0}}\int_0^1 \frac{1}{t^3}\int_0^t \frac{s^2}{\sqrt{1-s}}ds\,dt \\ &\le \frac{1}{\sqrt{b_0}}\int_0^1 \frac{1}{t^3}\frac{1}{\sqrt{1-t}}\int_0^t s^2\,ds\,dt \\ &= \frac{1}{3\sqrt{b_0}}\int_0^1 \frac{1}{\sqrt{1-t}}dt < \infty.\end{aligned}$$

We may now apply Theorem 2.1 to deduce the result.

Next we discuss the more general boundary value problem

$$\begin{cases}(p\,y')' + p(t)q(t)f(t,y,p\,y') = 0, \qquad 0<t<1\\ \lim_{t\to 0^+} p(t)y'(t) = y(1) = 0.\end{cases} \tag{2.30}$$

THEOREM 2.3. *Suppose the following conditions are satisfied*:

$$f:[0,1]\times(0,\infty)\times\mathbf{R}\to\mathbf{R} \text{ is continuous} \tag{2.31}$$

$$p\in C[0,1]\cap C^1(0,1) \qquad \text{with } p>0 \text{ on } (0,1) \tag{2.32}$$

$$q\in C(0,1) \qquad \text{with } q>0 \text{ on } (0,1) \tag{2.33}$$

let $n\in\{3,4,\dots\}$ *and associated with each* n *we have a constant* ρ_n *such that* ρ_n *is a nonincreasing sequence with* $\lim_{n\to\infty}\rho_n = 0$ *and such that for* $\frac{1}{n}\le t\le 1$ *we have* $p(t)q(t)f(t,\rho_n,0)\ge 0$ (2.34)

there exists a function $\alpha\in C[0,1]\cap C^2(0,1)$, $p\,\alpha'\in C[0,1]\cap C^1(0,1)$, *with* $\lim_{t\to 0^+}p(t)\alpha'(t) = \alpha(1) = 0$, $\alpha>0$ *on* [0,1), *such that for each* $n\in\{3,4,\dots\}$ *we have* $p(t)q(t)f(t,y,p(t)\alpha'(t)) + (p\,\alpha')'(t) > 0$ *for* $(t,y)\in\left[\frac{1}{n},1\right)\times\{y\in(0,\infty): y<\alpha(t)\}$ *and there exists* $\epsilon>0$ *with* $p(t)q(t)f\left(\frac{1}{n},y,z\right) + (p\,\alpha')'(t) > 0$ *for* $(t,y,z)\in\left(0,\frac{1}{n}\right)$ $\times\{y\in(0,\infty): y<\alpha(t)\}\times[p(t)\alpha'(t)-\epsilon,\, p(t)\alpha'(t)+\epsilon]$

(2.35)

for each $n\in\{3,4,\dots\}$ *there exists a function* $\beta_n\in C[0,1]\cap C^2(0,1)$, $p\,\beta_n'\in C[0,1]\cap C^1(0,1)$, *with* $\lim_{t\to 0^+}p(t)\beta_n'(t)\le 0$, $\beta_n(t)\ge\rho_n$ *for* $t\in[0,1]$, *and with* $p(t)q(t)f(t,\beta_n(t),p(t)\beta_n'(t)) + (p\,\beta_n')'(t)\le 0$ *for* $t\in\left[\frac{1}{n},1\right)$

(2.36)

for each $n\in\{3,4,\dots\}$ *we have* $p(t)q(t)f\left(\frac{1}{n},\beta_n(t),p(t)\beta_n'(t)\right)$ $+(p\,\beta_n')'(t)\le 0$ *for* $t\in\left(0,\frac{1}{n}\right)$ (2.37)

$$\max\left\{\sup_{t\in[0,1]} \beta_n(t) \colon n \in \{3,4,\dots\}\right\} < \infty,$$
$$\max\left\{\sup_{t\in[0,1]} |p(t)\beta_n'(t)| \colon n \in \{3,4,\dots\}\right\} < \infty \tag{2.38}$$

$|f(t,y,z)| \le [g(y)+h(y)]\psi(|z|)$ *on* $[0,1]\times(0,\infty)\times\mathbf{R}$ *with* $g>0$ *continuous and nonincreasing on* $(0,\infty)$, $h\ge 0$ *continuous on* $[0,\infty)$, $\frac{h}{g}$ *nondecreasing on* $(0,\infty)$, *and* $\psi>0$ *continuous on* $\mathbf{R}$

(2.39)

$$\int_0^1 p(t)q(t)g(\alpha(t))dt < \infty \qquad \textit{and}$$
$$\int_0^1 \frac{1}{p(s)}\int_0^s p(t)q(t)g(\alpha(t))dt\,ds < \infty \tag{2.40}$$

and

$$\left\{1+\frac{h(a_0)}{g(a_0)}\right\}\int_0^1 p(t)\,q(t)\,g(\alpha(t))\,dt < \int_0^\infty \frac{du}{\psi(u)}; \tag{2.41}$$

here $a_0 = \max\left\{\sup_{t\in[0,1]} \beta_n(t)\colon n\in\{3,4,\dots\}\right\}$. *Then* (2.30) *has a solution in* $y\in C[0,1]\cap C^2(0,1)$, $py'\in C[0,1]$ *with* $y(t)\ge\alpha(t)$ *for* $t\in[0,1]$.

Remark 2.6. In (2.34)–(2.38) we can replace $n\in\{3,4,\dots\}$ with $n\in\{n_0,n_0+1,\dots\}$ for some $n_0\in\{3,4,\dots\}$.

Remark 2.7. Note (2.40) guarantees that

$$\int_0^1 p(t)\,q(t)\,dt < \infty \qquad \text{and} \qquad \int_0^1 \frac{1}{p(s)}\int_0^s p(t)\,q(t)\,dt\,ds < \infty.$$

Proof. Fix $n\in\{3,4,\dots\}$. Choose $M>0$ so that

$$M > \max\left\{\max\left\{\sup_{t\in[0,1]} |p(t)\,\beta_n'(t)| \colon n\in\{3,4,\dots\}\right\},\ \sup_{t\in[0,1]} |p(t)\,\alpha'(t)|\right\} \tag{2.42}$$

and

$$\left\{1+\frac{h(a_0)}{g(a_0)}\right\}\int_0^1 p(t)\,q(t)\,g(\alpha(t))\,dt < \int_0^M \frac{du}{\psi(u)}. \tag{2.43}$$

Consider the boundary value problem

$$\begin{cases} (p\,y')' + p(t)\,q(t)\,f^{**}(t,\ y,\ p\,y') = 0, & 0<t<1 \\ \lim_{t\to 0^+} p(t)\,y'(t) = 0, & y(1) = \rho_n, \end{cases} \tag*{(2.44)^n}$$

where

$$f^{\star\star}(t,y,z)=\begin{cases} f\left(\frac{1}{n},\beta_n(t),z^\star\right)+r(\beta_n(t)-y), & y\geq\beta_n(t) \text{ and } 0\leq t\leq\frac{1}{n}\\ f(t,\beta_n(t),z^\star)+r(\beta_n(t)-y), & y\geq\beta_n(t) \text{ and } \frac{1}{n}\leq t\leq 1\\ f\left(\frac{1}{n},y,z^\star\right), & \rho_n\leq y\leq\beta_n(t) \text{ and } 0\leq t\leq\frac{1}{n}\\ f(t,y,z^\star), & \rho_n\leq y\leq\beta_n(t) \text{ and } \frac{1}{n}\leq t\leq 1\\ f(t,\rho_n,z^\star)+r(\rho_n-y), & y<\rho_n \text{ and } \frac{1}{n}\leq t\leq 1\\ f\left(\frac{1}{n},\rho_n,z^\star\right)+r(\rho_n-y), & y<\rho_n \text{ and } 0\leq t\leq\frac{1}{n}, \end{cases}$$

with

$$z^\star=\begin{cases} M, & z>M\\ z, & -M\leq z\leq M\\ -M, & z<-M \end{cases}$$

and $r:\mathbf{R}\to[-1,\ 1]$ is the radial retraction as defined in Theorem 2.1. It is easy to see, via Schauder's fixed point theorem (see [11]), that $(2.44)^n$ has a solution $y_n\in C[0,\ 1]\cap C^2(0,\ 1)$ with $py_n'\in C[0,\ 1]\cap C^1(0,\ 1)$. Essentially the same reasoning as in Theorem 2.1 (see below) guarantees that

$$y_n(t)\geq\rho_n \qquad \text{for } t\in[0,\ 1] \tag{2.45}$$

$$y_n(t)\leq\beta_n(t) \qquad \text{for } t\in[0,\ 1] \tag{2.46}$$

and

$$y_n(t)\geq\alpha(t) \qquad \text{for } t\in[0,\ 1]. \tag{2.47}$$

We will show (2.46) (the argument for (2.45) is similar) and (2.47). If (2.46) is false then $y_n-\beta_n$ would have a positive absolute maximum at say $t_0\in[0,\ 1)$. If $t_0\in(0,\ 1)$ then $p(y_n-\beta_n)'(t_0)=0$ and $(p(y_n-\beta_n)')'(t_0)\leq 0$. There are two cases to consider, namely $t_0\in\left[\frac{1}{n},\ 1\right)$ and $t_0\in\left(0,\ \frac{1}{n}\right)$.

Case (i). $t_0\in\left[\frac{1}{n},\ 1\right)$. Then since $y_n(t_0)>\beta_n(t_0)$ we have, using (2.36) and $\sup_{t\in[0,\,1]}|p(t)\,\beta_n'(t)|<M$, that

$$\begin{aligned}(p\,(y_n-\beta_n)')'(t_0) &= -q(t_0)\,p(t_0)\,f^{\star\star}(t_0,\ \beta_n(t_0),\ p(t_0)\,\beta_n'(t_0))-(p\,\beta_n')'(t_0)\\ &= -q(t_0)\,p(t_0)\,[f(t_0,\ \beta_n(t_0),\ p(t_0)\,\beta_n'(t_0))+r(\beta_n(t_0)\\ &\qquad -y_n(t_0))]-(p\,\beta_n')'(t_0)\\ &> 0,\end{aligned}$$

a contradiction.

Case (ii). $t_0 \in \left(0, \frac{1}{n}\right)$. Then (2.37) and $\sup_{t\in[0,1]} |p(t)\,\beta_n'(t)| < M$ give

$$\begin{aligned}(p(y_n - \beta_n)')'(t_0) &= -q(t_0)\,p(t_0)\left[f\left(\frac{1}{n}, \beta_n(t_0),\; p(t_0)\,\beta_n'(t_0)\right)\right.\\ &\qquad \left. + r(\beta_n(t_0) - y_n(t_0))\right] - (p\,\beta_n')'(t_0)\\ &> 0,\end{aligned}$$

a contradiction.

It remains to consider the case $t_0 = 0$. Now

$$\lim_{t\to 0^+} p(t)\,[y_n - \beta_n]'(t) = -\lim_{t\to 0^+} p(t)\,\beta_n'(t) \geq 0,$$

a contradiction unless $\lim_{t\to 0^+} p(t)\,\beta_n'(t) = 0$. Suppose $\lim_{t\to 0^+} p(t) \times \beta_n'(t) = 0$. Now $y_n(0) > \beta_n(0)$, $\lim_{t\to 0^+} p(t)\,y_n'(t) = \lim_{t\to 0^+} p(t)\,\beta_n'(t) = 0$, $f : [0, 1] \times (0, \infty) \times \mathbf{R} \to \mathbf{R}$ continuous, $\sup_{t\in[0,1]} |p(t)\,\beta_n'(t)| < M$ guarantees that there exists $\mu > 0$, $\mu < \frac{1}{n}$, with

$$y_n(s) > \beta_n(s), \qquad |p(s)\,y_n'(s)| < M \qquad \text{for } s \in (0, \mu),$$

and

$$\begin{aligned}&\left[f\left(\frac{1}{n}, \beta_n(s),\; p(s)\,y_n'(s)\right) - f\left(\frac{1}{n}, \beta_n(s), p(s)\,\beta_n'(s)\right)\right]\\ &\quad + r(\beta_n(s) - y_n(s)) < 0 \qquad \text{for } s \in (0, \mu).\end{aligned}$$

Thus for $t \in (0, \mu)$ we have

$$\begin{aligned}p\,(y_n - \beta_n)'(t) &= -\int_0^t [p(s)\,q(s)\,f^{\star\star}(s, y_n(s), p(s)\,y_n'(s)) + (p\,\beta_n')'(s)]\,ds\\ &= -\int_0^t \left[p(s)\,q(s)\left\{f\left(\frac{1}{n}, \beta_n(s), p(s)\,y_n'(s)\right)\right.\right.\\ &\qquad \left.\left. + r(\beta_n(s) - y_n(s))\right\} + (p\,\beta_n')'(s)\right] ds\\ &> -\int_0^t \left[p(s)\,q(s)\,f\left(\frac{1}{n}, \beta_n(s), p(s)\,\beta_n'(s)\right)\right.\\ &\qquad \left. + (p\,\beta_n')'(s)\right] ds \geq 0.\end{aligned}$$

That is, $p\,(y_n - \beta_n)'(t) > 0$ for $t \in (0, \mu)$, which is a contradiction. Thus (2.46) holds.

To see (2.47) suppose it is not true. Then $y_n - \alpha$ has a negative absolute minimum at say $t_1 \in [0, 1)$. If $t_1 \in (0, 1)$ then $p(y_n - \alpha)'(t_1) = 0$ and $(p\,(y_n - \alpha)')'(t_1) \geq 0$. There are two cases to consider, namely $t_1 \in \left[\frac{1}{n}, 1\right)$ and $t_1 \in \left(0, \frac{1}{n}\right)$.

Case (i). $t_1 \in \left[\frac{1}{n}, 1\right)$. Now $0 < y_n(t_1) < \alpha(t_1)$, $\rho_n \leq y_n(t_1) \leq \beta_n(t_1)$, $\sup_{t\in[0,1]} |p(t)\,\alpha'(t)| < M$, and (2.35) imply

$$\begin{aligned}(p\,(y_n-\alpha)')'(t_1) &= -[q(t_1)\,p(t_1)\,f^{\star\star}(t_1,\ y_n(t_1),\ p(t_1)\alpha'(t_1)) + (p\,\alpha')'(t_1)]\\ &= -[q(t_1)\,p(t_1)\,f(t_1,\ y_n(t_1),\ p(t_1)\,\alpha'(t_1)) + (p\,\alpha')'(t_1)]\\ &< 0,\end{aligned}$$

a contradiction.

Case (ii). $t_1 \in \left(0, \frac{1}{n}\right)$. Again (2.35) implies

$$\begin{aligned}(p\,(y_n-\alpha)')'(t_1) &= -[q(t_1)\,p(t_1)f^{\star\star}(t_1,\ y_n(t_1),\ p(t_1)\,\alpha'(t_1)) + (p\,\alpha')'(t_1)]\\ &= -[q(t_1)\,p(t_1)\,f\left(\frac{1}{n},\ y_n(t_1),\ p(t_1)\,\alpha'(t_1)\right) + (p\,\alpha')'(t_1)\\ &< 0,\end{aligned}$$

a contradiction.

It remains to consider the case $t_1 = 0$. Now $y_n(0) < \alpha(0)$ and $\sup_{t\in[0,1]} |p(t)\,\alpha'(t)| < M$ guarantees that there exists $\mu > 0$, $\mu < \frac{1}{n}$, with

$$y_n(s) < \alpha(s), \qquad |p(s)\,y_n'(s)| < M \qquad \text{for } s \in (0, \mu).$$

This with $\lim_{t\to 0^+} p\,(y_n-\alpha)'(t) = 0$ gives for $t \in (0, \mu)$,

$$\begin{aligned}p\,(y_n-\alpha)'(t) &= -\int_0^t [p(s)\,q(s)\,f^{\star\star}(s,\ y_n(s),\ p(s)\,y_n'(s)) + (p\,\alpha')'(s)]\,ds\\ &= -\int_0^t \left[p(s)\,q(s)f\left(\frac{1}{n}, y_n(s), p(s)\,y_n'(s)\right) + (p\,\alpha')'(s)\right] ds.\end{aligned}$$

In addition $\lim_{t\to 0^+} p(t)\,y_n'(t) = \lim_{t\to 0^+} p(t)\,\alpha'(t) = 0$ and (2.35) guarantees that there exists $\mu_0 \in (0, \mu)$ with

$$p(t)\,q(t)f\left(\frac{1}{n}, y_n(t), p(s)\,y_n'(t)\right) + (p\,\alpha')'(t) \;>\; 0$$

for $t \in (0,\ \mu_0]$. Thus for $t \in (0,\ \mu_0)$ we have $p\,(y_n-\alpha)'(t) < 0$, which is a contradiction. Thus (2.47) holds.

Remark 2.8. It is easy to check using the above type argument that $\alpha(t) \leq \beta_n(t)$ for $t \in [0,\ 1]$.

We next show

$$|p(t)\,y_n'(t)| \leq M \qquad \text{for } t \in [0,\ 1]. \tag{2.48}$$

Without loss of generality assume $p(t)\,y_n'(t) \not\leq M$ for some $t \in (0,1]$. This together with $\lim_{t\to 0^+} p(t)\,y_n'(t) = 0$ guarantees that there exists $t_1 \in [0,1)$, $t_2 \in (0,1)$, $t_1 < t_2$ with

$$\lim_{t\to t_1} p(t)\,y_n'(t) = 0, \qquad p(t_2)\,y_n'(t_2) = M$$

and $0 \leq p(s)\,y_n'(s) \leq M$ for s between t_1 and t_2. Now for $s \in (t_1, t_2)$ we have from (2.39) that

$$\begin{aligned}\pm(p\,y_n')'(s) &\leq q(s)\,p(s)\,g(y_n(s))\left\{1 + \frac{h(y_n(s))}{g(y_n(s))}\right\}\psi(|p(s)\,y_n'(s)|)\\ &\leq q(s)\,p(s)\,g(\alpha(s))\left\{1 + \frac{h(a_0)}{g(a_0)}\right\}\psi(p(s)\,y_n'(s));\end{aligned}$$

here $a_0 = \max\left\{\sup_{t\in[0,1]} \beta_n(t)\colon n \in \{3,4,\ldots\}\right\}$. Thus

$$\frac{(p\,y_n')'(s)}{\psi(p(s)\,y_n'(s))} \leq \left\{1 + \frac{h(a_0)}{g(a_0)}\right\} q(s)\,p(s)\,g(\alpha(s)) \qquad \text{for } s \in (t_1,\ t_2)$$

and so integration from t_1 to t_2 gives

$$\int_0^M \frac{du}{\psi(u)} \leq \left\{1 + \frac{h(a_0)}{g(a_0)}\right\} \int_0^1 q(s)\,p(s)\,g(\alpha(s))\,ds.$$

This contradicts (2.43), so (2.48) holds.

Notice

$$\{y_n\}_{n=3}^{\infty},\ \{p\,y_n'\}_{n=3}^{\infty} \text{ are bounded, equicontinuous families on } [0,\ 1].$$

To see equicontinuity notice (2.39) guarantees that

$$\begin{aligned}|(p\,y_n')'(s)| \leq &\left\{1 + \frac{h(a_0)}{g(a_0)}\right\}\left[\sup_{z\in[-M,M]} \psi(|z|)\right]\\ &\times p(s)\,q(s)\,g(\alpha(s)) \qquad \text{for } s \in (0,1)\end{aligned}$$

and

$$\begin{aligned}|y_n'(s)| \leq &\left\{1 + \frac{h(a_0)}{g(a_0)}\right\}\left[\sup_{z\in[-M,M]} \psi(|z|)\right]\\ &\times \frac{1}{p(s)} \int_0^s p(x)\,q(x)\,g(\alpha(x)) \qquad \text{for } s \in (0,1).\end{aligned}$$

The Arzela–Ascoli theorem guarantees the existence of a subsequence N_3 of integers and a function $y \in C^1[0,\ 1]$ (respectively $p\,y' \in C[0,\ 1]$) with y_n (respectively $p\,y_n'$) converging uniformly on $[0,\ 1]$ to y (respectively $p\,y'$) as $n \to \infty$ through N_3. Also $y(1) = 0$, $\lim_{t\to 0^+} p(t)\,y'(t) = 0$, $\alpha(t) \leq y(t) \leq a_0$ for $t \in [0,1]$ and $|p(t)\,y'(t)| \leq M$ for $t \in [0,1]$. Next fix $t \in (0,1)$ and let $m \in N_3$ be such that $\frac{1}{m} < t < 1$. Let $N_3^\star = \{n \in N_3 \colon n \geq m\}$. Note y_n, $n \in N_3^\star$, satisfies

$$y_n(t) = y_n(0) - \int_0^{\frac{1}{n}} \frac{1}{p(x)} \int_0^x p(s)\, q(s)\, f\left(\frac{1}{n}, y_n(s), p(s)\, y_n'(s)\right) ds\, dx$$
$$- \int_0^t \frac{1}{p(x)}\, \chi_{[\frac{1}{n}, t]}(x) \left[\int_0^{\frac{1}{n}} p(s)\, q(s)\, f\left(\frac{1}{n}, y_n(s), p(s)\, y_n'(s)\right) ds \right.$$
$$\left. + \int_0^x p(s)\, q(s)\, f(s, y_n(s), p(s)\, y_n'(s))\, \chi_{[\frac{1}{n}, x]}(s)\, ds \right] dx.$$

Let $n \to \infty$ through $N_3^\star$ to deduce that

$$y(t) = y(0) - \int_0^t \frac{1}{p(x)} \int_0^x p(s)\, q(s)\, f(s, y(s), p(s)\, y'(s))\, ds\, dx. \qquad \blacksquare$$

Remark 2.9. Obvious analogues of Remark 2.3, Remark 2.4, and Remark 2.5 are available for the boundary value problem (2.30). We leave the details to the reader.

The ideas in this section extend to other boundary data. For example, we could consider the Dirichlet problem

$$\begin{cases} (p\, y')' + p(t)\, \phi(t)\, f(t, y, p\, y') = 0, & 0 < t < 1 \\ y(0) = y(1) = 0, \end{cases} \tag{2.49}$$

where $p \in C[0, 1] \cap C^1(0, 1)$, $p > 0$ on $(0, 1)$, $\phi \in C(0, 1) \cap L^1[0, 1]$, $\phi > 0$ on $(0, 1)$, and $\int_0^1 \frac{ds}{p(s)} < \infty$. However, by making a standard change of variables (i.e., by using the Liouville transformation) one can reduce (2.49) to a problem of the form

$$\begin{cases} y'' + q(t)\, f(t, y, y') = 0, & 0<t<1 \\ y(0) = y(1) = 0. \end{cases} \tag{2.50}$$

Thus it is enough to consider the boundary value problem (2.50). We will now show how the theory in this section extends to the Dirichlet problem.

THEOREM 2.4. *Suppose the following conditions are satisfied*:

$$f\colon [0, 1] \times (0, \infty) \times \mathbf{R} \to \mathbf{R} \text{ is continuous} \tag{2.51}$$

$$q \in C(0, 1) \quad \text{with } q > 0 \text{ on } (0, 1) \quad \text{and } q \in L^1[0, 1] \tag{2.52}$$

let $n \in \{3, 4, \dots\}$ and associated with each n we have a constant ρ_n such that $\{\rho_n\}$ is a nonincreasing sequence with $\lim_{n\to\infty} \rho_n = 0$ and such that for $\frac{1}{n} \leq t \leq 1$ we have $q(t)\, f(t, \rho_n, 0) \geq 0$ (2.53)

$$\begin{aligned}&\textit{there exists a function } \alpha \in C^1[0,\,1] \cap C^2(0,\,1) \textit{ with}\\ &\alpha(0) = \alpha(1) = 0,\ \alpha > 0 \textit{ on } (0,\,1) \textit{ such that for each}\\ &n \in \{3, 4, \ldots\} \textit{ we have } q(t)\, f(t,\, y,\, \alpha'(t)) + \alpha''(t) > 0\\ &\textit{for } (t, y) \in \left[\tfrac{1}{n}, 1\right) \times \{y \in (0,\,\infty)\colon y < \alpha(t)\} \textit{ and}\\ &q(t)\, f\left(\tfrac{1}{n},\, y,\, \alpha'(t)\right) + \alpha''(t) > 0 \ \textit{ for } (t, y) \in \left(0,\, \tfrac{1}{n}\right)\\ &\times\{y \in (0,\,\infty)\colon y < \alpha(t)\}\end{aligned} \tag{2.54}$$

for each $n \in \{3, 4, \ldots\}$ *there exists a function* $\beta_n \in C^1[0,\,1] \cap C^2(0,\,1)$ *with* $\beta_n(t) \geq \rho_n$ *for* $t \in [0,\,1]$ *and* $q(t)\, f(t,\, \beta_n(t),\, \beta_n'(t)) + \beta_n''(t) \leq 0$ *for* $t \in \left[\frac{1}{n}, 1\right)$
(2.55)

$$\begin{aligned}&\textit{for each } n \in \{3, 4, \ldots\} \textit{ we have } q(t)\, f\left(\tfrac{1}{n},\, \beta_n(t),\, \beta_n'(t)\right)\\ &\quad +\beta_n''(t) \leq 0 \textit{ for } t \in \left(0,\, \tfrac{1}{n}\right)\end{aligned} \tag{2.56}$$

$$\begin{aligned}\max\left\{\sup_{t\in[0,\,1]} \beta_n(t) : n \in \{3, 4, \ldots\}\right\} &< \infty,\\ \max\left\{\sup_{t\in[0,\,1]} |\beta_n'(t)| : n \in \{3, 4, \ldots\}\right\} &< \infty\end{aligned} \tag{2.57}$$

$|f(t,\, y,\, z)| \leq [g(y) + h(y)]\, \psi(|z|)$ *on* $[0, 1] \times (0, \infty) \times \mathbf{R}$ *with* $g > 0$ *continuous and nonincreasing on* $(0,\,\infty)$, $h \geq 0$ *continuous on* $[0,\,\infty)$, $\frac{h}{g}$ *nondecreasing on* $(0,\,\infty)$, *and* $\psi > 0$ *continuous on* $\mathbf{R}$.
(2.58)

$$\int_0^1 q(t)\, g(\alpha(t))\, dt < \infty \tag{2.59}$$

and

$$\left\{1 + \frac{h(a_0)}{g(a_0)}\right\} \int_0^1 q(t)\, g(\alpha(t))\, dt \ < \int_0^\infty \frac{du}{\psi(u)}; \tag{2.60}$$

here $a_0 = \max\left\{\sup_{t\in[0,\,1]} \beta_n(t)\colon n \in \{3, 4, \ldots\}\right\}$. *Then* (2.50) *has a solution in* $y \in C^1[0,\,1] \cap C^2(0, 1)$ *with* $y(t) \geq \alpha(t)$ *for* $t \in [0,\,1]$.

Proof. Fix $n \in \{3, 4, \ldots\}$. Choose $M > 0$ so that

$$M \ > \ \max\left\{\max\left\{\sup_{t\in[0,\,1]} |\beta_n'(t)| : n \in \{3, 4, \ldots\}\right\},\ \sup_{t\in[0,\,1]} |\alpha'(t)|\right\} \tag{2.61}$$

and

$$\left\{1 + \frac{h(a_0)}{g(a_0)}\right\} \int_0^1 q(t)\, g(\alpha(t))\, dt \ < \int_0^M \frac{du}{\psi(u)}. \tag{2.62}$$

Consider the boundary value problem

$$\begin{cases} y'' + q(t)f^{\star\star}(t, y, y') = 0, & 0 < t < 1 \\ y(0) = y(1) = \rho_n, \end{cases} \tag{2.63^n}$$

where $f^{\star\star}$ is as in Theorem 2.3. It is easy to see using Schauder's fixed point theorem (see [11]) that $(2.63)^n$ has a solution $y_n \in C^1[0,1] \cap C^2(0,1)$. We first show

$$y_n(t) \geq \rho_n \qquad \text{for } t \in [0,1]. \tag{2.64}$$

Suppose (2.64) is not true. Then $y_n - \rho_n$ has a negative absolute minimum at say $t_0 \in (0,1)$, in which case $y_n'(t_0) = 0$ and $y_n''(t_0) \geq 0$. However,

$$\begin{aligned} y_n''(t_0) &= -q(t_0)f^{\star\star}(t_0, y_n(t_0), 0) \\ &= \begin{cases} -q(t_0)\,[f(t_0, \rho_n, 0) + r(\rho_n - y_n(t_0))] & \text{if } \frac{1}{n} \leq t_0 < 1 \\ -q(t_0)\,\left[f\left(\frac{1}{n}, \rho_n, 0\right) + r(\rho_n - y_n(t_0))\right] & \text{if } 0 \leq t_0 \leq \frac{1}{n}, \end{cases} \end{aligned}$$

i.e., $y_n''(t_0) < 0$, a contradiction. Thus (2.64) holds. Next we show

$$y_n(t) \leq \beta_n(t) \qquad \text{for } t \in [0,1]. \tag{2.65}$$

If (2.65) is not true then $y_n - \beta_n$ would have a positive absolute maximum at say $t_0 \in (0,1)$. Essentially the same reasoning as in Theorem 2.3 (see Cases (i) and (ii)) will lead to a contradiction. Thus (2.65) is true. In particular $y_n(t) \leq a_0 = \max\left\{\sup_{t\in[0,1]} \beta_n(t) : n \in \{3,4,\ldots\}\right\}$ for $t \in [0,1]$. Next we obtain a sharper lower bound on y_n, namely we will show

$$y_n(t) \geq \alpha(t) \qquad \text{for } t \in [0,1]. \tag{2.66}$$

Suppose (2.66) is not true. Then $y_n - \alpha$ has a negative absolute minimum at say $t_1 \in (0,1)$. Essentially the same reasoning as in Theorem 2.3 (see Cases (i) and (ii)) will lead to a contradiction. Thus (2.66) is true.

Essentially the same reasoning as in Theorem 2.3 also yields

$$|y_n'(t)| \leq M \qquad \text{for } t \in [0,1]. \tag{2.67}$$

Notice (2.65), (2.66), (2.67), together with

$$|y_n''(s)| \leq \left\{1 + \frac{h(a_0)}{g(a_0)}\right\}\left[\sup_{z\in[-M,M]} \psi(|z|)\right] q(s)g(\alpha(s)) \qquad \text{for } s \in (0,1),$$

and (2.59) guarantee that

$$\left\{y_n^{(j)}\right\}_{n=3}^{\infty} (j = 0, 1) \text{ is a bounded, equicontinuous family on } [0,1]. \tag{2.68}$$

The Arzela–Ascoli theorem guarantees the existence of a subsequence N_3 of integers and a function $y \in C^1[0,1]$ with $y_n^{(j)}$ converging uniformly on

$[0, 1]$ to $y^{(j)}$ $(j = 0, 1)$ as $n \to \infty$ through N_3. Note $y(0) = y(1) = 0$, $\alpha(t) \le y(t) \le a_0$ for $t \in [0, 1]$ and $|y'(t)| \le M$ for $t \in [0, 1]$. Next fix $t \in (0, 1)$ (without loss of generality assume $t \ne \frac{1}{2}$) and let $m \in \{3, 4, \ldots\}$ be such that $\frac{1}{m} < t < 1$. Let $N_3^\star = \{n \in N_3 : n \ge m\}$. Note $y_n, n \in N_3^\star$, satisfies

$$\begin{aligned} y_n(t) &= y_n\left(\frac{1}{2}\right) + y_n'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) + \int_{\frac{1}{2}}^t (s - t)\, q(s)\, f^\star(s, y_n(s), y_n'(s))\, ds \\ &= y_n\left(\frac{1}{2}\right) + y_n'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) + \int_{\frac{1}{2}}^t (s - t)\, q(s)\, f(s, y_n(s), y_n'(s))\, ds. \end{aligned}$$

Let $n \to \infty$ through $N_3^\star$ to obtain

$$y(t) = y\left(\frac{1}{2}\right) + y'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) + \int_{\frac{1}{2}}^t (s - t)\, q(s)\, f(s, y(s), y'(s))\, ds.$$

We can do this argument for each $t \in (0, 1)$, so $y''(t) + q(t)f(t, y(t), y'(t)) = 0$ for $t \in (0, 1)$ and $y \in C^2(0, 1)$. ∎

Remark 2.10 Obvious analogues of Remark 2.2, Remark 2.3, and Remark 2.4 are available for the boundary value problem (2.50). We leave the details to the reader.

Remark 2.11 Notice (2.59) and (2.60) can be replaced by

$$\exists\, r,\ 1 \le r < \infty, \text{ and } p \text{ (the conjugate to } r\text{) with } \int_0^1 [q(t)]^p\, dt < \infty \tag{2.69}$$

$$\int_0^{a_0} [g(u)]^r\, du < \infty \tag{2.70}$$

and

$$\left\{1 + \frac{h(a_0)}{g(a_0)}\right\}\left(\int_0^1 [q(t)]^p dt\right)^{\frac{1}{p}}\left(\int_0^{a_0} [g(u)]^r du\right)^{\frac{1}{r}} < \int_0^\infty \frac{u^{\frac{1}{r}}}{\psi(u)}\, du \tag{2.71}$$

and the result in Theorem 2.4 is again true (note if $p = \infty$ we denote $\|q\|_{L^\infty}$ by $\left(\int_0^1 [q(t)]^p dt\right)^{1/p}$). To see this, notice in this case we choose $M > 0$ so that (2.61) and

$$\left\{1 + \frac{h(a_0)}{g(a_0)}\right\}\left(\int_0^1 [q(t)]^p\, dt\right)^{\frac{1}{p}}\left(\int_0^{a_0} [g(u)]^r du\right)^{\frac{1}{r}} < \int_0^M \frac{u^{\frac{1}{r}}}{\psi(u)}\, du \tag{2.72}$$

hold. Essentially the same reasoning as in Theorem 2.4 establishes the proof. The only difference in the proof is showing (2.67). Suppose this is false. Without loss of generality assume $y_n'(t) \not\le M$ for some $t \in [0, 1]$.

There exists $t_1, t_2 \in (0,1)$ with $y_n'(t_1) = 0$, $y_n'(t_2) = M$ and $0 \le y_n'(s) \le M$ for s between t_1 and t_2. Without loss of generality assume $t_1 < t_2$. Notice

$$y_n''(s) \le \left\{1 + \frac{h(a_0)}{g(a_0)}\right\} q(s)\, g(y_n(s))\, \psi(y_n'(s)) \qquad \text{for } s \in (t_1, t_2)$$

and so

$$\frac{[y_n'(s)]^{\frac{1}{r}}\, y_n''(s)}{\psi(y_n'(s))} \le \left\{1 + \frac{h(a_0)}{g(a_0)}\right\} q(s)\, g(y_n(s))\, [y_n'(s)]^{\frac{1}{r}} \qquad \text{for } s \in (t_1, t_2).$$

Integrate from t_1 to t_2 to obtain

$$\int_0^M \frac{u^{\frac{1}{r}}}{\psi(u)} du \le \left\{1 + \frac{h(a_0)}{g(a_0)}\right\} \left(\int_0^1 [q(t)]^p dt\right)^{\frac{1}{p}} \left(\int_0^{a_0} [g(u)]^r du\right)^{\frac{1}{r}}.$$

This contradicts (2.72). Note also that instead of (2.70) and (2.71) we could have assumed

$$\int_0^1 [q(s)\, g(\alpha(s))]^p\, ds < \infty \tag{2.73}$$

and

$$\left\{1 + \frac{h(a_0)}{g(a_0)}\right\} \left(\int_0^1 [q(s)\, g(\alpha(s))]^p\, ds\right)^{\frac{1}{p}} [a_0]^{\frac{1}{r}} < \int_0^\infty \frac{u^{\frac{1}{r}}}{\psi(u)} du. \tag{2.74}$$

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